Introduction to Hilbert C^*-modules, II

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Set $N = \{z \in H \otimes A^{**} : \langle z, z \rangle = 0\}$. Then $(H \otimes A^{**})/N$ becomes a pre-Hilbert A^{**}-module containing H as an A-submodule. Denote by $H^\sim$ the Hilbert A^{**}-module $((H \otimes A^{**})/N)^-)^\sharp$. It is self-dual, i.e., $(H^\sim)^\sharp = H^\sim$. Moreover, $B(H^\sim) = L(H^\sim)$ is an W^*-algebra.

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$$\langle T(\sum_i h_i \otimes a_i), \sum_j x_j \otimes b_j \rangle = \sum_{i,j} a_i^* [T(h_i)(x_j)] b_j.$$

Then T becomes an element in $B(((H \otimes A^{**})/N)^{-}, H^{\sim})$. So T then extends to a map in $B(H^{\sim})$ with $\|\tilde{T}\| = \|T\|$. Moreover such extension is unique.

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Since E is full and $Px(\sum_{i=1}^m \langle y_i, w_i \rangle \in E$ for all $y_i, w_i \in E$, we conclude from the above inequalities that $Px \in E$ for all $x \in H$. Therefore $P \in B(H)$ and $H = (I - P)H \oplus E$. This completes the proof.

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In other words, we are search a map $\tilde{T}$ with $\|\tilde{T}\| = \|T\|$ such that the following commutative diagram commutes:

$$\begin{array}{ccc} H_2 & & \\ \uparrow & \searrow^{\tilde{T}} & \\ H_1 & \longrightarrow_{T} & H \end{array}$$

Let C_1 be category whose objects are Hilbert A-modules and morphisms are contractive module maps with adjoints. We would like to identify those injective objects.

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Theorem *2.9 Let A be a C^*-algebra and H be a Hilbert A-module. Then H is injective in the category C_1 if and only if H is orthogonal complementary.*

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$$H_2 = \ker T \oplus |T_\lambda|(H_2).$$

Furthermore, T_λ is one-to-one on $|T_\lambda|(H_2)$ and maps $|T_\lambda|(H_2)$ onto $0 \oplus H$. By Cor. 2.3, $|T_\lambda|(H_2) \cong H$.

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It should be noted that for the implications (1) $\Rightarrow$ (2) (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) we do not need to assume that A is σ-unital.

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